Ultrarelativistic Decoupling Transformation for Generalized Dirac Equations

J. H. Noble and U. D. Jentschura*
Department of Physics, Missouri University of Science and Technology, Rolla, Missouri 65409, USA

The Foldy–Wouthuysen transformation is known to uncover the nonrelativistic limit of a generalized Dirac Hamiltonian, lending an intuitive physical interpretation to the effective operators within Schrödinger–Pauli theory. We here discuss the opposite, ultrarelativistic limit which requires the use of a fundamentally different expansion where the leading kinetic term in the Dirac equation is perturbed by the mass of the particle and other interaction (potential) terms, rather than vice versa. The ultrarelativistic decoupling transformation is applied to free Dirac particles (in the Weyl basis) and to high-energy tachyons, which are faster-than-light particles described by a fully Lorentz-covariant equation. The effective gravitational interactions are found. For tachyons, the dominant gravitational interaction term in the high-energy limit is shown to be attractive, and equal to the leading term for subluminal Dirac particles (tardyons) in the high-energy limit.

PACS numbers: 11.10.-z, 03.65.Pm, 04.25.dg, 95.36.+x, 98.80.-k

I. INTRODUCTION

The Foldy–Wouthuysen transformation [1] is an established method, used to calculate the nonrelativistic limit of effective Hamiltonians describing spin-1/2 particles. The procedure has been applied with good effect to the Dirac–Coulomb Hamiltonian [2, 3], uncovering the fine-structure terms for atomic levels, notably, the zitterbewegung term, and the Russell–Saunders (spin-orbit) coupling (Thomas precession). Recently, the analogue of the Russell–Saunders coupling in a gravitational field (the Fokker precession, see Refs. [4–6]) has been recovered from the gravitationally coupled Dirac Hamiltonian, which is referred to as the Dirac–Schwarzschild Hamiltonian [5].

The Foldy–Wouthuysen program, in its original form [1], is inherently perturbative in nature. In a typical case, the structure of a generalized Dirac Hamiltonian is given as $\vec{\alpha} \cdot \vec{p} + \beta\, m + \delta H$ (in the standard Dirac representation of the Clifford algebra, see Appendix A). Here, the "dominant" term is taken as $\beta\, m$, where β is the 4×4 Dirac β matrix, $\vec{\alpha} \cdot \vec{p}$ is the kinetic operator ($\vec{\alpha}$ is the vector of Dirac α matrices, and $\vec{p}$ is the momentum operator), and δH contains the potential terms. One then expands about a Dirac particle "at rest", with the dominant term given by the "rest mass" term βm. The Foldy–Wouthuysen procedure then uncovers the leading nonrelativistic kinetic term $\vec{\alpha} \cdot \vec{p} \to \vec{p}^{\,2}/(2m) + \dots$ and transforms the potential terms δH into a form where the operators acquire an intuitive physical interpretation. At some risk to oversimplification, one can say that the Foldy–Wouthuysen transformation applies to the regime $|\vec{\alpha} \cdot \vec{p}| \ll |\beta m|$, and $|\delta H| \ll |\beta m|$.

In some cases, such as for a free Dirac particle [2], it is sometimes possible to perform the transformation without any perturbative expansion in the momenta or other expansion parameters. There have been attempts to generalize the idea of a nonperturbative method to more general Hamiltonians, and a set of interesting identities have been derived Ref. [7]. However, the alternative Foldy–Wouthuysen transformation [7] suffers from an explicit breaking of the parity symmetry in the transformation, which involves the fifth current, and is known to produce spurious parity-breaking terms in a number of applications, e.g., to the Dirac–Coulomb Hamiltonian (for an overview, see Refs. [6, 8, 9]). In general, nonperturbative methods (in the momenta of the Dirac particles) can only be applied when considerable additional information is available for a specific Hamiltonian under investigation, and when additional approximations are made, such as the neglecting terms of second order and higher in the field strengths [see Eq. (21) of Ref. [10]].

To the best of our knowledge, the opposite perturbative expansion, namely, perturbation theory of a Dirac Hamiltonian about the ultrarelativistic limit, has not yet been considered in the literature; it is the subject of the current paper. This expansion has to follow a fundamentally different paradigm; in the ultrarelativistic limit, mass terms and potential terms are suppressed in comparison to the kinetic term; the expansion is valid in the regime $|\beta m| \ll |\vec{\alpha} \cdot \vec{p}|$, and $|\delta H| \ll |\vec{\alpha} \cdot \vec{p}|$. Ultrarelativistic particles are best described in the helicity basis [Chap. 23 of Ref. [11]], while in fact, the solutions of the free Dirac equation approach those of the Weyl equation in the massless limit (see Chap. 2.4.3 on p. 87 of Ref. [3]). The Weyl equation describes massless spin-1/2 particles, which transform under the fundamental $(\frac{1}{2}, 0)$ representation of the Lorentz group and travel exactly at the speed of light (these are the "neutrinos in the original standard model").

We here investigate the ultrarelativistic decoupling transformation with a special emphasis on the gravitational coupling of a particle to a central gravitational field. To this end, in Sec. II, we briefly recall the underlying covariant formalism, distinguishing the case of a "normal" (subluminal) Dirac particle from a particle described by the tachyonic Dirac equation [4–6]. The latter equation describes faster-than-light particles, still

* ulj@mst.edu

in a fully Lorentz-covariant formalism [12]. The ultrarelativistic limit specifically is relevant to tachyons because these particles cannot travel slower than light; they remain superluminal upon Lorentz transformation [13–17]. In the ultrarelativistic limit, the particle's speed approaches the light cone and the influence of tardyonic as well as tachyonic mass terms are suppressed in comparison to the kinetic terms. In Sec. III, the ultrarelativistic decoupling transformation is applied to gravitationally coupled tardyonic and tachyonic particles. Conclusions are reserved for Sec. IV.

II. FREE PARTICLES

A. Free Tardyonic Transformation

In principle it is well known that the Weyl equation, which describes a massless spin-1/2 particle, splits into two equations, describing a left-handed and a right-handed spinor (see Chap. 23 of Ref. [11] and p. 87 of Ref. [3]),

$$\mathrm{i}\,\partial_t \psi_L = H_L\,\psi_L\,, \qquad H_L = -\vec{\sigma}\cdot\vec{p}\,, \tag{1}$$

$$\mathrm{i}\,\partial_t \psi_R = H_R\,\psi_R\,, \qquad H_R = \vec{\sigma}\cdot\vec{p}\,, \tag{2}$$

The Weyl equations break parity; a left-handed spinor transforms into a right-handed solution under the parity operation. However, it is well known that the Dirac equation, whose bispinor solutions are constructed by stacking the helicity spinors on top of each other, conserves parity [18].

The massless equation, in turn, corresponds to the ultrarelativistic limit for a massive Dirac particle; we would thus expect that the Dirac equation splits into two equations, describing left- and right-handed Weyl spinors, in this limit. Thus, if we are to recover the massless (Weyl) limit, plus corrections, from the Dirac equation, then we need to necessarily invoke a parity-breaking transformation. We start from the free Dirac Hamiltonian

$$H_{\rm FD} = \vec{\alpha}\cdot\vec{p} + \beta\,m = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \tag{3}$$

and invoke the following unitary, parity-breaking transformation

$$U = \frac{1}{\sqrt{2}}\,(1-\beta\,\gamma^5)\,, \qquad U^{-1} = U^{\rm T} = \frac{1}{\sqrt{2}}\,(1+\beta\,\gamma^5)\,, \tag{4}$$

which transforms $H_{\rm FD}$ into $\mathbf{H}_{\rm FD} = U\,H_{\rm FD}\,U^{-1}$,

$$\mathbf{H}_{\rm FD} = -\beta\vec{\Sigma}\cdot\vec{p} + \gamma^5\,m = \begin{pmatrix} -\vec{\sigma}\cdot\vec{p} & m \\ m & \vec{\sigma}\cdot\vec{p} \end{pmatrix} \tag{5}$$

The initial rotation with the U matrix corresponds to a change of the basis of the Dirac algebra, into the so-called Weyl basis (see Appendix A). The mass terms are now off-diagonal and we may try to eliminate them by an ultrarelativistic decoupling (ultrarelativistic Foldy–Wouthuysen) transformation. To this end we define the energy operator

$$\mathcal{E} = -\vec{\Sigma}\cdot\vec{p}\,, \tag{6}$$

and the transformation (see Sec 4.2 of Ref. [2]) and Sec. 2.2.4 of Ref. [3])

$$S_{\rm FD} = -\mathrm{i}\,\beta\,\gamma^5\,\frac{m}{\mathcal{E}}\Theta\,, \qquad S_{\rm FD}^{+} = S_{\rm FD}\,. \tag{7}$$

so that the unitary transformation $U_{\rm FD}$ becomes

$$U_{\rm FD} = \exp(\mathrm{i}\,S_{\rm FD}) = \cos\left(\frac{m\,\Theta}{|\vec{p}|}\right) + \beta\,\gamma^5\,\frac{|\vec{p}|}{\mathcal{E}}\sin\left(\frac{m\,\Theta}{|\vec{p}|}\right) \tag{8}$$

Choosing Θ so that

$$\cos\left(2\frac{m\,\Theta}{|\vec{p}|}\right) = \frac{|\vec{p}|}{\sqrt{\vec{p}^{\,2}+m^2}}\,, \tag{9a}$$

$$\sin\left(2\frac{m\,\Theta}{|\vec{p}|}\right) = \frac{m}{\sqrt{\vec{p}^{\,2}+m^2}}\,, \tag{9b}$$

one finally obtains

$$\mathcal{H}_{\rm FD} = U_{\rm FD}\,\mathbf{H}_{\rm FD}\,U_{\rm FD}^{-1} = \frac{\mathcal{E}}{|\vec{p}|}\,\sqrt{\vec{p}^{\,2}+m^2}\,. \tag{10}$$

In explicit (2×2)-matrix subform,

$$\begin{aligned}\mathcal{H}_{\rm FD} &= -\beta\,\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}\,\sqrt{\vec{p}^{\,2}+m^2} \\ &= \begin{pmatrix} -\dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\,\sqrt{\vec{p}^{\,2}+m^2} & 0 \\ 0 & \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\,\sqrt{\vec{p}^{\,2}+m^2} \end{pmatrix}\,,\end{aligned} \tag{11}$$

it becomes clear that the separation into a left-handed (upper diagonal) and a right-handed (lower diagonal) Hamiltonian has been achieved.

The eigenstates of the Hamiltonian (10) fulfill

$$\mathrm{i}\,\partial_t\psi_i(t,\vec{r}) = \mathcal{H}_{\rm FD}\,\psi_i(t,\vec{r})\,, \qquad i = 1,2,3,4\,. \tag{12a}$$

The first two solutions can be written as

$$\psi_1(t,\vec{r}) = \begin{pmatrix} a_-(\vec{k}) \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}E\,t+\mathrm{i}\,\vec{k}\cdot\vec{r}}\,, \qquad H_{\rm FD}\,\psi_1 = E\,\psi_1\,, \tag{12b}$$

$$\psi_2(t,\vec{r}) = \begin{pmatrix} a_-(\vec{k}) \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}E\,t-\mathrm{i}\,\vec{k}\cdot\vec{r}}\,, \qquad H_{\rm FD}\,\psi_2 = -E\,\psi_2\,. \tag{12c}$$

The physical momentum is $\vec{k}$, and the helicity eigenvalue is negative for both solutions, $\vec{\Sigma}\cdot\hat{k}\,\psi_{1,2} = -\psi_{1,2}$, and

$E = \sqrt{\vec{k}^2 + m^2}$ (here, $\hat{k}$ is the unit vector in the $\vec{k}$ direction). The solution ψ_2 describes an antiparticle. The two solutions of right-handed helicity are

$$\psi_3(t, \vec{r}) = \begin{pmatrix} 0 \\ a_+(\vec{k}) \end{pmatrix} \mathrm{e}^{-\mathrm{i} E\, t + \mathrm{i}\, \vec{k}\cdot\vec{r}}\,, \qquad H_{\rm FD}\, \psi_3 = E\, \psi_3\,, \tag{12d}$$

$$\psi_4(t, \vec{r}) = \begin{pmatrix} 0 \\ a_+(\vec{k}) \end{pmatrix} \mathrm{e}^{\mathrm{i} E\, t - \mathrm{i}\, \vec{k}\cdot\vec{r}}\,, \qquad H_{\rm FD}\, \psi_4 = -E\, \psi_4\,, \tag{12e}$$

The helicity is positive for these two solutions, $\vec{\Sigma}\cdot\hat{k}\,\psi_{3,4} = \psi_{3,4}$, with ψ_4 describing an antiparticle. The eigenvalues of the $\mathcal{E}$ operator for $\psi_{1,2,3,4}$ are $E, -E, -E, E$, respectively. If we apply the formalism to a Dirac neutrino, them ψ_1 would describe a left-handed neutrino, ψ_2 would describes a left-handed antineutrino, whereas ψ_3 and ψ_4 would describe right-handed neutrinos and right-handed antineutrinos, respectively. For completeness, we recall the form of the helicity spinors [3],

$$a_+(\vec{k}) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{\mathrm{i}\,\varphi} \end{pmatrix}, \tag{13a}$$

$$a_-(\vec{k}) = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \mathrm{e}^{-\mathrm{i}\,\varphi} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{13b}$$

where θ and φ are the polar and azimuthal angles of the $\vec{k}$ vector.

B. Free Tachyonic Transformation

As we have just shown, one may accomplish an exact diagonalization (in spinor space) of the free Dirac Hamiltonian using the ultrarelativistic decoupling transformation. However, one might counter argue that this result is in principle familiar: An exact diagonalization can also be accomplished using the Foldy–Wouthuysen transformation (in its original form) for the free Dirac Hamiltonian (see Sec. 4.2 of Ref [2]). The ultrarelativistic transform leads to a form which asymptotically is equal to the Weyl Hamiltonian (helicity basis) of a massless particle, as it should be (in the ultrarelativistic limit). Here, we shall make the point that, unlike the original Foldy–Wouthuysen transform, which can only be applied to tardyons, the ultrarelativistic decoupling can also be used for Lorentz-invariant tachyons [12, 19, 20], whose velocity remains superluminal upon Lorentz transformation [13–17].

A few general remarks on tachyonic spin-1/2 particles might be in order. The tachyonic neutrino hypothesis remains one of the driving forces behind the study of the tachyonic Dirac equation [12]. The algbraic structures underlying the tachyonic spin-1/2 equation have recently been studied in greater depth (see Refs. [20–24] and references therein). Pertinent potentially relevant astrophysical observations have recently been recorded in Refs. [25–28]; other theoretical studies concern Dirac equations with Lorentz-violating terms which can lead to superluminal propagation for neutrinos [29, 30]. The tachyonic Dirac Hamiltonian has recently been identified as a pseudo-Hermitian ("γ^5–Hermitian") Hamiltonian in Ref. [20]. Independent of the phenomenological relevance of the concept of tachyons, the current section of our paper, and Sec. III B demonstrate that it is possible to uncover the leading terms of generalized pseudo-Hermitian [31–35] Dirac Hamiltonians in the ultrarelativistic limit using the relativistic decoupling transformation.

The accepted generalized Dirac Hamiltonian for a free tachyonic Dirac particle is given as [12, 19, 20]

$$H_{\rm TD} = \vec{\alpha}\cdot\vec{p} + \beta\,\gamma^5\, m\,, \tag{14}$$

which is γ^5-Hermitian, $H_{\rm TD} = \gamma^5\, H^+_{\rm TD}\,\gamma^5$. We then follow the same procedure outlined in Sec. II A, and begin by performing the initial rotation U [see Eq. (4)], giving us

$$\mathbf{H}_{\rm TD} = U\, H_{\rm TD}\, U^{-1} = \beta\,\mathcal{E} + \beta\,\gamma^5\, m\,. \tag{15}$$

The Hamiltonian $\mathbf{H}_{\rm TD}$ is β-Hermitian, i.e., $\mathbf{H}_{\rm TD} = \beta\,\mathbf{H}^+_{\rm TD}\,\beta$. Here, β is the Dirac β matrix which takes the role of the γ^5 matrix in the Weyl representation (see Appendix A). The β–Hermitian operator $S_{\rm TD}$ in this case reads as

$$S_{\rm TD} = -\mathrm{i}\beta\,\frac{\beta\,\gamma^5\, m}{\mathcal{E}}\Theta = -\mathrm{i}\,\gamma^5\,\frac{m}{\mathcal{E}}\Theta\,, \qquad S_{\rm TD} = \beta\, S^+_{\rm TD}\,\beta\,. \tag{16}$$

The transformation

$$U_{\rm TD} = \exp(\mathrm{i}S_{\rm TD}) = \cosh\left(\frac{m}{|\vec{p}|}\Theta\right) + \gamma^5\,\frac{|\vec{p}|}{\mathcal{E}}\sinh\left(\frac{m}{|\vec{p}|}\Theta\right) \tag{17}$$

fulfills the identity

$$U^+_{\rm TD}\,\beta\, U_{\rm TD} = \exp(\mathrm{i}S_{\rm TD})\beta\exp(\mathrm{i}S_{\rm TD}) = \beta\,, \tag{18}$$

i.e., it is β-unitary. It therefore conserves the $\mathcal{PT}$-symmetric scalar product $\langle\psi|\beta|\phi\rangle$. Choosing [redefining, see Eq. (9)] Θ so that

$$\cosh\left(2\frac{m\,\Theta}{|\vec{p}|}\right) = \frac{|\vec{p}|}{\sqrt{\vec{p}^{\,2} - m^2}}\,, \tag{19a}$$

$$\sinh\left(2\frac{m\,\Theta}{|\vec{p}|}\right) = \frac{m}{\sqrt{\vec{p}^{\,2} - m^2}}\,, \tag{19b}$$

one obtains

$$\mathcal{H}_{\rm TD} = U_{\rm TD}\,\mathbf{H}_{\rm TD}\, U^{-1}_{\rm TD} = \beta\frac{\mathcal{E}}{|\vec{p}|}\sqrt{\vec{p}^{\,2} - m^2}\,. \tag{20}$$

This amounts to the exact ultrarelativistic decoupling transformation of the free tachyonic Hamiltonian, in the

helicity ("Weyl") basis which has been shown to lead to a very efficient description of the tachyonic bispinor solutions [19, 24, 36]. The Taylor series expansion of $\mathcal{H}_{\rm TD}$ gives rise to the terms

$$\begin{aligned}\mathcal{H}_{\rm TD} &\approx \beta \left(\mathcal{E} - \frac{m^2}{2\mathcal{E}} - \frac{m^4}{8\mathcal{E}^3} \right) \\ &= -\beta \vec{\Sigma} \cdot \vec{p} \left(1 - \frac{m^2}{2|\vec{p}|^2} - \frac{m^4}{8|\vec{p}|^4} \right) , \end{aligned} \tag{21}$$

which are the correction terms in the high-energy limit. For the tachyonic case, the eigenstates of the Hamiltonian (20) are still given by Eq. (12), but one has to replace $E = \sqrt{\vec{k}^2 + m^2} \to \sqrt{\vec{k}^2 - m^2}$, in the sense of the tachyonic dispersion relation.

III. TRANSFORMATION WITH GRAVITATIONAL COUPLING

A. Gravitational Tardyonic Transformation

The study of the gravitationally coupled Dirac equation, for massless particles, was initiated by the question of how the neutrinos (assumed by symmetry to be strictly massless in the Original Standard Model) interact with gravitational fields [37]. We follow this route and start from the gravitationally coupled Dirac–Schwarzschild Hamiltonian [5]

$$H_{\rm DS} = \frac{1}{2} \left\{ \vec{\alpha} \cdot \vec{p}, 1 - \frac{r_s}{r} \right\} + \beta\, m \left(1 - \frac{r_s}{2r} \right) . \tag{22}$$

After the initial transformation into the Weyl basis, one finds for $\mathbf{H}_{\rm DS} = U\, H_{\rm DS}\, U_1^{-1}$ [see Eq. (4)]

$$\begin{aligned}\mathbf{H}_{\rm DS} &= \frac{\beta}{2} \left\{ \mathcal{E}, 1 - \frac{r_s}{r} \right\} + \gamma^5\, m \left(1 - \frac{r_s}{2r} \right) \\ &= \begin{pmatrix} -\frac{1}{2} \left\{ \vec{\sigma} \cdot \vec{p}, 1 - \frac{r_s}{r} \right\} & m \left(1 - \frac{r_s}{2r} \right) \\ m \left(1 - \frac{r_s}{2r} \right) & \frac{1}{2} \left\{ \vec{\sigma} \cdot \vec{p}, 1 - \frac{r_s}{r} \right\} \end{pmatrix} . \end{aligned} \tag{23}$$

In order to proceed with the ultrarelativistic decoupling transformation, we identify the odd part $\mathcal{O}_{\rm DS}$ of $\mathbf{H}_{\rm DS}$ and define

$$\mathcal{O}_{\rm DS} = \gamma^5\, m \left(1 - \frac{r_s}{2r} \right) , \quad S_{\rm DS} = -\mathrm{i} \frac{\beta}{4} \left\{ \mathcal{O}_{\rm DS}, \frac{1}{\mathcal{E}} \right\} , \tag{24}$$

where $S_{\rm DS}$ is Hermitian. The unitary transformation $U_{\rm DS} = \exp(\mathrm{i}\, S_{\rm DS})$ is applied to calculate $\mathcal{H}_{\rm DS} = U_{\rm DS}\, \mathbf{H}_{\rm DS}\, U_{\rm DS}^{-1}$, perturbatively,

$$\mathbf{H}'_{\rm DS} \approx \mathbf{H}_{\rm DS} + \frac{\mathrm{i}^1}{1!} [S_{\rm DS}, \mathbf{H}_{\rm DS}] + \frac{\mathrm{i}^2}{2!} [S_{\rm DS}, [S_{\rm DS}, \mathbf{H}_{\rm DS}]] + \dots \tag{25}$$

which is a series of nested commutators, as with the classic Foldy–Wouthuysen transformation [1]. In the following, we carry the calculation to first order in the Schwarzschild radius r_s (first order in G) and keep inverse powers of $\mathcal{E}$ up to order $1/\mathcal{E}$.

It is advantageous to write the Hamiltonian (23) as

$$\mathbf{H}_{\rm DS} = \beta\, \mathcal{E} - \frac{\beta}{2} \left\{ \mathcal{E}, \frac{r_s}{r} \right\} + \mathcal{O}_{\rm DS} . \tag{26}$$

The first commutator is given as

$$\begin{aligned}[S_{\rm DS}, \mathbf{H}_{\rm DS}] = {}& [S_{\rm DS}, \beta\, \mathcal{E}] - \left[S_{\rm DS}, \frac{\beta}{2} \left\{ \mathcal{E}, \frac{r_s}{r} \right\} \right] \\ & + [S_{\rm DS}, \mathcal{O}_{\rm DS}] . \end{aligned} \tag{27}$$

Let us investigate the first commutator $[S_{\rm DS}, \beta\, \mathcal{E}]$, for which one finds after a somewhat tedious calculation,

$$[S_{\rm DS}, \beta \mathcal{E}] = \mathrm{i} \mathcal{O}_{\rm DS} + \frac{\mathrm{i}}{4} \frac{1}{\mathcal{E}} [\mathcal{E}, [\mathcal{E}, \mathcal{O}_{\rm DS}]] \frac{1}{\mathcal{E}} , \tag{28}$$

where the double commutator is proportional to a three-dimensional Dirac-δ function plus a spin orbit coupling term,

$$[\mathcal{E}, [\mathcal{E}, \mathcal{O}_{\rm DS}]] = -2\pi\, r_s\, \delta^{(3)}(\vec{r}) - r_s \frac{\vec{\Sigma} \cdot \vec{L}}{r^3} , \tag{29}$$

which is of order unity in the expansion in inverse powers of $\mathcal{E}$. Despite the fact that the double commutator has two instances of the operator $\mathcal{E}$, the commutators ensure that these instances of $\mathcal{E}$ act *only* on $\mathcal{O}_{\rm DS}$, and *not* on the reference state wave function, which would otherwise generate inverse powers of $\mathcal{E}$. For the operator $\mathcal{E}$ (or the inverse thereof) to be the "dominant term", it must operate on a wave function describing a high-energy particle. Thus

$$\frac{1}{\mathcal{E}} [\mathcal{E}, [\mathcal{E}, \mathcal{O}_{\rm DS}]] \frac{1}{\mathcal{E}} = \mathcal{O} \left(\frac{1}{\mathcal{E}^2} \right) \to 0 . \tag{30}$$

Alternatively, one may observe that, when using the Weyl free–spinors given in Eq. (12) as reference states, the the expectation values of both the Dirac–δ function and the spin–orbit coupling term ($\vec{\Sigma} \cdot \vec{L}/r^3$) vanish for both diagonal as well as off-diagonal matrix elements. In conclusion, to the order relevant for our investigation, we can replace

$$[S_{\rm DS}, \beta\, \mathcal{E}] \to \mathrm{i} \mathcal{O}_{\rm DS} , \tag{31}$$

in our approximation. This relation ensures the odd terms will be canceled out to the first order in $\mathcal{O}_{\rm DS}$ when calculating the transformed Hamiltonian $\mathcal{H}_{\rm DS}$ according to Eq. (25). One also establishes that

$$\left[S_{\rm DS}, \frac{\beta}{2} \left\{ \mathcal{E}, \frac{r_s}{r} \right\} \right] = \mathrm{i} \gamma^5 m \frac{r_s}{r} , \tag{32a}$$

$$[S_{\rm DS}, \mathcal{O}_{\rm DS}] = \mathrm{i}\, \beta \left(-\frac{m^2}{\mathcal{E}} + \frac{1}{2} m^2 \left\{ \frac{1}{\mathcal{E}}, \frac{r_s}{r} \right\} \right) , \tag{32b}$$

so that the first commutator becomes

$$[S_{\rm DS}, \mathbf{H}_{\rm DS}] = \mathrm{i}\left(\mathcal{O}_{\rm DS} + \gamma^5 m \frac{r_s}{r} - \beta\frac{m^2}{\mathcal{E}} + \frac{\beta\, m^2}{2}\left\{\frac{1}{\mathcal{E}}, \frac{r_s}{r}\right\}\right). \tag{33}$$

The double commutator is then

$$[S_{\rm DS}, [S_{\rm DS}, \mathbf{H}_{\rm DS}]] = \mathrm{i}\left([S_{\rm DS}, \mathcal{O}_{\rm DS}] + \left[S_{\rm DS}, \gamma^5 m \frac{r_s}{r}\right] - \left[S_{\rm DS}, \beta\frac{m^2}{\mathcal{E}}\right] + \left[S, \frac{\beta\, m^2}{2}\left\{\frac{1}{\mathcal{E}}, \frac{r_s}{r}\right\}\right]\right), \tag{34}$$

where the first term is known from Eq. (32b). The other relevant commutators are

$$\left[S_{\rm DS}, \gamma^5 m \frac{r_s}{r}\right] = \mathrm{i}\beta\frac{m^2}{2}\left\{\frac{1}{\mathcal{E}}, \frac{r_s}{r}\right\}, \tag{35a}$$

$$-\left[S_{\rm DS}, \beta\frac{m^2}{\mathcal{E}}\right] = \mathcal{O}\left(\frac{1}{\mathcal{E}^2}\right) \to 0\,, \tag{35b}$$

$$\left[S_{\rm DS}, \frac{1}{2}\,\beta\, m^2\left\{\frac{1}{\mathcal{E}}, \frac{r_s}{r}\right\}\right] = \mathcal{O}\left(\frac{1}{\mathcal{E}^2}\right) \to 0\,. \tag{35c}$$

We then have

$$[S_{\rm DS}, [S, \mathbf{H}_{\rm DS}]] = \beta\frac{m^2}{\mathcal{E}} - \beta m^2\left\{\frac{1}{\mathcal{E}}, \frac{r_s}{r}\right\}, \tag{36}$$

where again we neglect higher-order terms. Because $S_{\rm DS}$ carries an inverse power of $\mathcal{E}$, we can neglect the triple commutator,

$$[S_{\rm DS}, [S_{\rm DS}, [S_{\rm DS}, \mathbf{H}_{DS}]]] = \mathcal{O}\left(\frac{1}{\mathcal{E}^2}\right) \to 0\,. \tag{37}$$

Thus

$$\begin{aligned}\mathbf{H}'_{\rm DS} &= \mathbf{H}_{\rm DS} + \mathrm{i}\,[S_{\rm DS}, \mathbf{H}_{\rm DS}] + \frac{\mathrm{i}^2}{2!}\,[S_{\rm DS}, [S_{\rm DS}, \mathbf{H}_{\rm DS}]] \\ &= \beta\left(\mathcal{E} + \frac{m^2}{2\mathcal{E}} - \frac{1}{2}\left\{\mathcal{E}, \frac{r_s}{r}\right\}\right) + \mathcal{O}'_{\rm DS}\,,\end{aligned} \tag{38}$$

where

$$\mathcal{O}'_{\rm DS} = -\gamma^5\, m\,\frac{r_s}{r}\,. \tag{39}$$

The second iteration of the transform with

$$S'_{\rm DS} = -\mathrm{i}\frac{\beta}{4}\left\{\mathcal{O}'_{\rm DS}, \frac{1}{\mathcal{E}}\right\}, \qquad U'_{\rm DS} = \exp(\mathrm{i}S'_{\rm DS})\,, \tag{40}$$

will serve only to eliminate the remaining odd term. Thus, the final result for $\mathcal{H}_{\rm DS} = U'_{\rm DS}\,\mathbf{H}'_{\rm DS}\,U'^{-1}_{\rm DS}$ reads as

$$\mathcal{H}_{\rm DS} = \beta\left(\mathcal{E} + \frac{m^2}{2\mathcal{E}} - \frac{1}{2}\left\{\mathcal{E}, \frac{r_s}{r}\right\}\right) \tag{41}$$

for a gravitationally coupled high-energy Dirac particle.

B. Gravitational Tachyonic Transformation

We start from the tachyonic, gravitationally coupled (TG) Dirac Hamiltonian derived in Appendix C,

$$H_{\rm TG} = \frac{1}{2}\left\{\vec{\alpha}\cdot\vec{p}, \left(1 - \frac{r_s}{r}\right)\right\} + \beta\,\gamma^5\, m\,\left(1 - \frac{r_s}{2r}\right). \tag{42}$$

The initial rotation into the Weyl basis of the Dirac algebra using the transformation U defined in Eq. (4) leads to the Hamiltonian $\mathbf{H}_{\rm TG} = U\,H_{\rm TG}\,U^{-1}$, which reads as

$$\begin{aligned}\mathbf{H}_{\rm TG} &= \frac{\beta}{2}\left\{\mathcal{E}, 1 - \frac{r_s}{r}\right\} + \beta\,\gamma^5\, m\,\left(1 - \frac{r_s}{2r}\right) \\ &= \begin{pmatrix} -\frac{1}{2}\{\vec{\sigma}\cdot\vec{p}, 1 - \frac{r_s}{r}\} & m\,\left(1 - \frac{r_s}{2r}\right) \\ -m\,\left(1 - \frac{r_s}{2r}\right) & \frac{1}{2}\{\vec{\sigma}\cdot\vec{p}, 1 - \frac{r_s}{r}\} \end{pmatrix},\end{aligned} \tag{43}$$

where $\mathcal{E}$ has been defined in Eq. (6). One identifies the odd part of the Hamiltonian $\mathbf{H}_{\rm TG}$ and writes

$$\mathcal{O}_{\rm TG} = \beta\,\gamma^5\, m\left(1 - \frac{r_s}{2r}\right), \quad S_{\rm TG} = -\mathrm{i}\frac{\beta}{4}\left\{\mathcal{O}_{\rm TG}, \frac{1}{\mathcal{E}}\right\}. \tag{44}$$

The β-unitary transformation $U_{\rm TG} = \exp(\mathrm{i}\,S_{\rm TG})$ is applied to calculate $\mathbf{H}'_{\rm TG} = U_{\rm TG}\,\mathbf{H}_{\rm TG}\,U^{-1}_{\rm TG}$, perturbatively,

$$\mathbf{H}'_{\rm TG} = \mathbf{H}_{\rm TG} + \frac{\mathrm{i}^1}{1!}[S_{\rm TG}, \mathbf{H}_{\rm TG}] + \frac{\mathrm{i}^2}{2!}[S_{\rm TG}, [S_{\rm TG}, \mathbf{H}_{\rm TG}]] + \ldots \tag{45}$$

in full analogy with the Dirac–Schwarzschild Hamiltonian. After a somewhat tedious calculation, neglecting (as before) the Dirac-δ and spin–orbit coupling terms, one finds

$$[S_{\rm TG}, \beta\mathcal{E}] = \mathrm{i}\mathcal{O}_{\rm TG}\,, \tag{46a}$$

$$\left[S_{\rm TG}, \frac{\beta}{2}\left\{\mathcal{E}, \frac{r_s}{r}\right\}\right] = \mathrm{i}\;\beta\,\gamma^5 m\,\frac{r_s}{r}\,, \tag{46b}$$

$$[S_{\rm TG}, \mathcal{O}_{\rm TG}] = \mathrm{i}\beta\, m^2\frac{1}{\mathcal{E}} - \mathrm{i}\beta\,\frac{m^2}{2}\left\{\frac{1}{\mathcal{E}}, \frac{r_s}{r}\right\}. \tag{46c}$$

The first commutator becomes

$$[S_{\rm TG}, \mathbf{H}_{\rm TG}] = \mathrm{i}\left(\mathcal{O}_{\rm TG} - \beta\gamma^5 m\frac{r_s}{r} + \beta\frac{m^2}{\mathcal{E}} - \frac{\beta m^2}{2}\left\{\frac{1}{\mathcal{E}}, \frac{r_s}{r}\right\}\right). \tag{47}$$

The double nested commutator is

$$[S_{\rm TG}, [S_{\rm TG}, H]] = \mathrm{i}\left([S_{\rm TG}, \mathcal{O}_{\rm TG}] - \left[S_{\rm TG}, \beta\,\gamma^5 m\frac{r_s}{r}\right] + \left[S_{\rm TG}, \beta\frac{m^2}{\mathcal{E}}\right] - \left[S_{\rm TG}, \beta\,\frac{m^2}{2}\left\{\frac{1}{\mathcal{E}}, \frac{r_s}{r}\right\}\right]\right), \tag{48}$$

The last two commutators are of order $1/\mathcal{E}^2$ and can therefore be neglected. With the help of the result

$$\left[S, \beta\gamma^5\, m\,\frac{r_s}{r}\right] = \mathrm{i}\frac{\beta m^2}{2}\left\{\frac{1}{\mathcal{E}}, \frac{r_s}{r}\right\}. \tag{49}$$

and with Eq. (46c), one finds

$$[S_{\rm TG}, [S_{\rm TG}, \mathbf{H}_{\rm TG}]] = -\beta\, m^2 \frac{1}{\mathcal{E}} + \beta\, m^2 \left\{ \frac{1}{\mathcal{E}}, \frac{r_s}{r} \right\} . \tag{50}$$

Thus,

$$\begin{aligned} \mathbf{H}'_{\rm TG} &= \mathbf{H}_{\rm TG} + {\rm i}\; [S_{\rm TG}, \mathbf{H}_{\rm TG}] + \frac{{\rm i}^2}{2!}\; [S_{\rm TG}, [S_{\rm TG}, \mathbf{H}_{\rm TG}]] \\ &= \beta \left(\mathcal{E} + \frac{m^2}{2\mathcal{E}} - \frac{1}{2} \left\{ \mathcal{E}, \frac{r_s}{r} \right\} \right) + \mathcal{O}'_{\rm TG} , \end{aligned} \tag{51}$$

where

$$\mathcal{O}'_{\rm TG} = \beta \gamma^5\, m \frac{r_s}{r} . \tag{52}$$

A second transformation with

$$S'_{\rm TG} = -{\rm i} \frac{\beta}{4} \left\{ \mathcal{O}'_{\rm TG}, \frac{1}{\mathcal{E}} \right\} , \qquad U'_{\rm TG} = \exp({\rm i} S'_{\rm TG}) , \tag{53}$$

eliminates $\mathcal{O}'_{\rm TG}$, and we obtain the following final result for $\mathcal{H}_{\rm TG} = U'_{\rm TG}\, \mathbf{H}'_{\rm TG}\, {U'}^{-1}_{\rm TG}$,

$$\mathcal{H}_{\rm TG} = \beta \left(\mathcal{E} - \frac{m^2}{2\mathcal{E}} - \left\{ \frac{\mathcal{E}}{2}, \frac{r_s}{r} \right\} \right) . \tag{54}$$

It differs from the result given in Eq. (41) only in the sign of the kinetic term $-m^2/(2\mathcal{E})$, due to the tachyonic dispersion relation.

IV. CONCLUSIONS

We have studied the ultrarelativistic decoupling transformation for the free Dirac equation (Sec. II A), and for the free tachyonic Dirac equation (Sec. II B). These transformations lead to a full separation of the Dirac equation in the helicity basis. Unlike the exact Foldy–Wouthuysen transformation, which transforms the free Dirac Hamiltonian into the form $\beta \sqrt{\vec{p}^2 + m^2}$ (see Ref. [2]), the ultrarelativistic transformation leads to a separation in the helicity basis, with the transformed Hamiltonian being proportional to $(-\beta\, \vec{\Sigma} \cdot \vec{p})$ [see Eqs. (10) and (20)]. The eigenstates of this Hamiltonian are naturally obtained in the helicity basis [see Eq. (12)] and are formally identical (upon a redefinition of the energy parameter E) to the eigenstates of the massless Dirac equation (see Chap. 2.4.3 on p. 87 of Ref. [3]). The latter eigenstates are known to transform under the fundamental $(\frac{1}{2}\,, 0)$ representation of the Lorentz group; the "helicity of the massless spinors does not flip upon a Lorentz transformation". This observation is intimately linked to the fact that massless Dirac spinors describe particles which always move at the speed of light; it is impossible to "overtake" the particle, which otherwise leads to a helicity flip (see Ref. [24]).

The initial unitary transformation U given in Eq. (4) transforms the Dirac equation into the Weyl basis (see Appendix A), which is naturally identified as the *ultrarelativistic basis* for the description of the Dirac algebra: Namely, the Dirac $\vec{\alpha}$ matrices are replace, in the Weyl basis, by matrices $(-\beta\, \vec{\Sigma} \cdot \vec{p})$, which are diagonal in the (2×2)-spinor space and describe the Hamiltonian for a massless Dirac particle. The Dirac and Weyl representations of the Clifford algebra are complementary: In the Dirac basis, the "dominant term" in the Hamiltonian is $\beta\, m$, and the odd (off-diagonal) kinetic terms $\vec{\alpha} \cdot \vec{p}$ are eliminated by the Foldy–Wouthuysen transformation. In the Weyl basis, the kinetic term $(-\beta\, \vec{\Sigma} \cdot \vec{p})$ is diagonal ("dominates in the ultrarelativistic limit), while the off-diagonal mass term $\gamma^5\, m$ needs to be eliminated by the ultrarelativistic decoupling transformation.

With gravitational coupling in a central, static field, the dominant attractive term is found to be described by the replacement $\mathcal{E} \to \left\{ \frac{\mathcal{E}}{2}, 1 - \frac{r_s}{r} \right\}$ in Eqs. (41) and (54), where $\mathcal{E}$ is the energy operator defined in Eq. (6). This replacement holds both for tardyons and tachyons and is a consequence of the structure of the Dirac–Schwarzschild Hamiltonian given in Eqs. (22) and (42). Namely, the dominant interaction in the high-energy limit is given by the anticommutator correction $\frac{1}{2} \left\{ \vec{\alpha} \cdot \vec{p}, \left(1 - \frac{r_s}{r}\right) \right\}$ in the original Hamiltonians (before ultrarelativistic decoupling) given in Eqs. (22) and (42). The somewhat surprising observation that high-energy tachyons are attracted by gravitational fields finds a natural explanation in the energy-mass equivalence, and in the observation that both tachyons as well as tardyons travel at speeds very close to the speed of light in the high-energy limit. Indeed, the only difference in the effective high-energy Hamiltonians (41) and (54) lies in the sign of the kinetic term $\pm m^2/(2\mathcal{E})$, which is due to the changes in the dispersion relation for tardyons as opposed to tachyons. Higher-order corrections to the gravitational coupling are discussed in Appendix D.

The ultrarelativistic decoupling transformation should find applications beyond the description of gravitational interactions, for highly relativistic particles subject to electromagnetic fields, and further applications to "nearly massless" electrons in graphene can be imagined (here, the "speed of light" is replaced by the Fermi velocity v_F, and dislocation potentials are added "by hand", see Ref. [38]).

Appendix A: Dirac and Weyl Basis

In the Dirac basis, we have

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} , \qquad \beta = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} . \tag{A1}$$

The γ matrices in the Dirac basis are $\gamma^0 = \beta$ and

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} , \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} . \tag{A2}$$

In the Weyl basis, we have

$$\vec{\alpha}_{\rm W} = \left(\begin{array}{cc} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{array} \right) , \qquad \beta_{\rm W} = \left(\begin{array}{cc} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{array} \right) . \quad \text{(A3)}$$

We define the vector of $\vec{\alpha}_{\rm W}$ matrices so that the "upper" solution includes the left-handed neutrino, whereas the "lower" spinor contains the right-handed Dirac antineutrino [see Eq. (12)]. The γ matrices in the Weyl basis are $\gamma^0_{\rm W} = \beta_{\rm W}$ and

$$\vec{\gamma}_{\rm W} = \left(\begin{array}{cc} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{array} \right) , \qquad \gamma^5_{\rm W} = \left(\begin{array}{cc} -\mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{array} \right) , \quad \text{(A4)}$$

so that $\gamma^5_{\rm W} = -\beta$. We notice that $\vec{\alpha}_{\rm W} = \beta_{\rm W} \, \vec{\gamma}_{\rm W}$ and also $\vec{\alpha}_{\rm W} = -\beta \, \vec{\Sigma}$ where

$$\vec{\Sigma} = \left(\begin{array}{cc} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{array} \right) . \quad \text{(A5)}$$

The vector of Pauli spin matrices is denoted as $\vec{\sigma}$. Note that some authors define the vector $\vec{\gamma}_{\rm W}$ with the opposite sign, which also reverses the sign of $\gamma^5_{\rm W} = {\rm i} \, \gamma^0_{\rm W} \, \gamma^1_{\rm W} \gamma^2_{\rm W} \, \gamma^3_{\rm W}$.

Incidentally, the Coulomb coupling identifies particles (which are attracted) and antiparticles (which are repulsed). It is interesting to verify whether the interpretation is preserved under the transformation to the Weyl representation. We start from the Hamiltonian

$$H_C = \vec{\alpha} \cdot \vec{p} - \frac{Z\alpha}{r} , \quad \text{(A6)}$$

which describes a massless particles in a Coulomb field (here, Z is the nuclear charge number, while α is the fine-structure constant). Transformation to the Weyl representation is accomplished by the rotation

$$\begin{aligned} \mathbf{H}_C &= U \, H_C \, U^{-1} = -\beta \, \vec{\Sigma} \cdot \vec{p} - \frac{Z\alpha}{r} \\ &= \left(\begin{array}{cc} -\vec{\sigma} \cdot \vec{p} - \frac{Z\alpha}{r} & 0 \\ 0 & \vec{\sigma} \cdot \vec{p} - \frac{Z\alpha}{r} \end{array} \right) . \end{aligned} \quad \text{(A7)}$$

A comparison with Eq. (12) reveals that states with positive unperturbed energy (positive eigenvalue of the operator $-\vec{\sigma} \cdot \vec{p}$ for the upper spinor and positive eigenvalue of $\vec{\sigma} \cdot \vec{p}$ for the lower spinor) are attracted by the Coulomb field. By contrast, states with negative unperturbed energy (negative eigenvalue of the operator $-\vec{\sigma} \cdot \vec{p}$ for the upper spinor and negative eigenvalue of $\vec{\sigma} \cdot \vec{p}$ for the lower spinor) are repulsed by the Coulomb field.

Appendix B: Operators

We wish to explore the application of the operator $1/\mathcal{E} = -1/(\vec{\Sigma} \cdot p)$ to a reference state wave function. To this end, we assume that $f = f(\vec{r})$ is a test function, and we defined the Fourier transform $\mathcal{F}$ and Fourier back-transform $\mathcal{F}^{-1}$ as follows,

$$(\mathcal{F} f)(\vec{k}) = \int {\rm d}^3 r \, \exp(-{\rm i} \vec{k} \cdot \vec{r}) \, f(\vec{r}) , \quad \text{(B1)}$$

$$(\mathcal{F}^{-1} F)(\vec{r}) = \int \frac{{\rm d}^3 k}{(2\pi)^3} \, \exp({\rm i} \vec{k} \cdot \vec{r}) \, f(\vec{k}) . \quad \text{(B2)}$$

One first multiplies the operator

$$1/\mathcal{E} \to -1/(\vec{\Sigma} \cdot \vec{k}) = -\frac{\vec{\Sigma} \cdot \vec{k}}{\vec{k}^2} \quad \text{(B3)}$$

in Fourier space and then transforms back to coordinate space,

$$\begin{aligned} \left(\frac{1}{\mathcal{E}} f \right) (\vec{r}) &= \left[\mathcal{F}^{-1} \left(-\frac{\vec{\Sigma} \cdot \vec{k}}{\vec{k}^2} \, (\mathcal{F} \, f) \, (\vec{k}) \right) \right] (\vec{r}) \\ &= -\int \frac{{\rm d}^3 k}{(2\pi)^3} \int {\rm d}^3 r' \, \frac{\vec{\Sigma} \cdot \vec{k}}{\vec{k}^2} {\rm e}^{{\rm i} \vec{k} \cdot (\vec{r} - \vec{r}\,')} \, f(\vec{r}\,') . \end{aligned} \quad \text{(B4)}$$

For a reference state with a special value $\vec{k}_s$ of the wave vector [see Eq. (12)],

$$f_s(\vec{r}) = \left. \psi_1(\vec{r}) \right|_{\vec{k} \to \vec{k}_s, t=0} = \left(\begin{array}{c} a_-(\vec{k}_s) \\ 0 \end{array} \right) \exp\left({\rm i} \vec{k}_s \cdot \vec{r} \right) , \quad \text{(B5)}$$

one has

$$(\mathcal{F} \, f_s)(\vec{k}) = \left(\begin{array}{c} a_-(\vec{k}_s) \\ 0 \end{array} \right) \delta^{(3)}(\vec{k} - \vec{k}_s) \quad \text{(B6)}$$

and so

$$-\frac{\vec{\Sigma} \cdot \vec{k}}{\vec{k}^2} (\mathcal{F} \, f_s)(\vec{k}) = \frac{1}{|\vec{k}_s|} \left(\begin{array}{c} a_-(\vec{k}_s) \\ 0 \end{array} \right) \delta^{(3)}(\vec{k} - \vec{k}_s) , \quad \text{(B7)}$$

whose Fourier backtransform is

$$\left(\frac{1}{\mathcal{E}} f_s \right) (\vec{r}) = \frac{1}{|\vec{k}_s|} \left(\begin{array}{c} a_-(\vec{k}_s) \\ 0 \end{array} \right) \exp\left({\rm i} \vec{k}_s \cdot \vec{r} \right) . \quad \text{(B8)}$$

This corresponds to the naive result that we obtain when interpreting the $\mathcal{E}$ operator as an energy operator and applying it to the eigenstates of the free Hamiltonian, given in Eq. (12).

Appendix C: Formalism for Gravitational Coupling

We here follow the conventions used in Refs. [4, 5] for the flat–space and curved–space Dirac gamma matrices. Specifically, the flat–space and curved–space Dirac gamma matrices are distinguished in this Appendix using a tilde ($\widetilde{\gamma}$) and an overline ($\overline{\gamma}$) respectively. We draw inspiration from the book [39] and denote indices related to a local Lorentz frame ("anholonomic basis") with capital

Latin indices $A, B, C, \ldots = 0, 1, 2, 3$. The curved-space Dirac gamma matrices $\overline{\gamma}^\mu$ satisfy the condition that

$$\{\overline{\gamma}^\mu(x), \overline{\gamma}^\nu(x)\} = 2\overline{g}^{\mu\nu}(x)\,, \tag{C1}$$

where $\overline{g}^{\mu\nu}(x)$ is the curved-space-time metric. The $\overline{\gamma}^\mu(x)$ are expressed in terms of the flat-space Dirac $\widetilde{\gamma}$ matrices $\widetilde{\gamma}^A$ as follows,

$$\overline{\gamma}^\mu(x) = e^\mu_A\, \widetilde{\gamma}^A\,, \qquad \overline{\gamma}_\mu(x) = e^A_\mu\, \widetilde{\gamma}_A\,, \tag{C2}$$

where the e^μ_A are the coefficients which relate the locally flat Lorentz frame to the global space-time coordinates (the "vierbein"). Greek indices $\mu, \nu, \rho, \ldots = 0, 1, 2, 3$ denote the global coordinates. Latin indices starting with $i, j, k, \ldots = 1, 2, 3, \ldots$ are reserved for "spatial" global coordinates, which leaves $I, J, K, \cdots = 1, 2, 3, \ldots$ for spatial coordinates in the anholonomic basis. This notation addresses some ambiguities which could otherwise result from other approaches [37, 40–46]. For example, unless the Dirac matrices are distinguished by overlining or tildes, the expression γ^1 could be associated with a flat-space matrix $\widetilde{\gamma}^{I=1}$ or with a curved-space matrix $\overline{\gamma}^{i=1}$. We use the "West–Coast" convention for the flat-space metric, which we denote as $\eta^{AB} = \eta_{AB} = \mathrm{diag}(1, -1, -1, -1)$. The curved-space metric is recovered as

$$\eta_{AB} = \frac{1}{2}\{\widetilde{\gamma}_A, \widetilde{\gamma}_B\}\,, \tag{C3}$$

$$\overline{g}_{\mu\nu}(x) = \frac{1}{2}\{\overline{\gamma}_\mu(x), \overline{\gamma}_\nu(x)\} = e^A_\mu\, e^B_\nu\, \eta_{AB}\,, \tag{C4}$$

$$\overline{g}^{\mu\nu}(x) = \frac{1}{2}\{\overline{\gamma}^\mu(x), \overline{\gamma}^\nu(x)\} = e^\mu_A\, e^\nu_B\, \eta^{AB}\,. \tag{C5}$$

For the curved–space metric around a gravitational center, we use the isotropic Schwarzschild metric in the Eddington reparameterization [47], i.e.

$$\overline{g}_{\mu\nu} = \mathrm{diag}(w^2, -v^2, -v^2, -v^2)\,, \tag{C6}$$

$$\overline{g}^{\mu\nu} = \mathrm{diag}(w^{-2}, -v^{-2}, -v^{-2}, -v^{-2})\,, \tag{C7}$$

$$w = \frac{1 - \dfrac{r_s}{4r}}{1 + \dfrac{r_s}{4r}}\,, \qquad v = \left(1 + \frac{r_s}{4r}\right)^2\,. \tag{C8}$$

For the Schwarzschild geometry, the vierbein coefficients read as follows,

$$e^0_\mu = \delta^0_\mu\, w\,, \qquad e^A_\mu = \delta^A_\mu\, v\,, \tag{C9}$$

$$e^\mu_0 = \frac{\delta^\mu_0}{w}\,, \qquad e^\mu_A = \frac{\delta^\mu_A}{v}\,. \tag{C10}$$

Here, $\delta^\mu_A = \delta^A_\mu$ denotes the Kronecker-δ (which is of course equal to unity for the two indices being equal and zero otherwise).

In full analogy with the case of a "normal" massive Dirac particle (see Refs. [4, 5]), we write the Dirac action for a tachyon in curved spacetime as

$$S = \int d^4x\, \sqrt{-\det \overline{g}(x)} \times \overline{\psi}(x)\, \overline{\gamma}^5(x)\, \left(\frac{\mathrm{i}}{2}\gamma^\rho(x) \overleftrightarrow{\nabla}_\rho - \overline{\gamma}^5(x)\, m\right) \psi(x)\,, \tag{C11}$$

where $\overline{\psi}(x)\, \overline{\gamma}^5(x)$ takes the role of the "tachyonic adjoint" (see Ref. [36]) and

$$\nabla_\rho = \partial_\rho - \Gamma_\rho(x)\,, \tag{C12}$$

$$\Gamma_\mu(x) = \frac{\mathrm{i}}{4}\, \omega^{AB}_\mu(x)\, \widetilde{\sigma}_{AB}\,, \qquad \widetilde{\sigma}_{AB} = \frac{\mathrm{i}}{2}\, [\widetilde{\gamma}_A, \widetilde{\gamma}_B]\,, \tag{C13}$$

$$\omega^{AB}_\nu(x) = e^A_\mu\, \nabla_\nu\, e^{\mu B} = e^A_\mu\, \partial_\nu\, e^{\mu B} + e^A_\mu\, \Gamma^\mu_{\nu\lambda}\, e^{\lambda B}\,. \tag{C14}$$

Here, the $\Gamma^\mu_{\nu\lambda}$ are the Christoffel symbols, and the $\omega^{AB}_\nu(x)$ are known as the Ricci rotation coefficients. Under a spinor Lorentz transformation with generators $\Omega^{AB}(x)$,

$$\psi'(x') = S(\Lambda(x))\, \psi(x) = \exp\left(-\frac{\mathrm{i}}{4}\, \Omega^{AB}(x)\, \widetilde{\sigma}_{AB}\right)\, \psi(x) \tag{C15}$$

we have covariance, i.e., $\nabla'_\nu\, \psi'(x) = \nabla'_\nu\, [S(\Lambda(x))\psi(x)] = (\partial_\nu - \Gamma'_\mu)\, [S(\Lambda(x))\psi(x)] = S(\Lambda(x))\, \nabla_\nu\, \psi(x)$, where the tranformed Ricci rotation coefficients $\Gamma'_\mu = \frac{\mathrm{i}}{4}\, {\omega'}^{AB}_\mu(x)\, \widetilde{\sigma}_{AB}$ are calculated with respect to the transformed local coordinates.

The curved-space fifth current $\overline{\gamma}^5(x)$ needs to be clarified. Adopting Eq. (18) of Ref. [37] for West–Coast sign conventions, one finds

$$\widetilde{\gamma}^5(x) = \frac{\mathrm{i}}{4!}\, \frac{\epsilon_{\mu\nu\rho\delta}}{\sqrt{-\det g(x)}} \widetilde{\gamma}^\mu(x)\, \widetilde{\gamma}^\nu(x)\, \widetilde{\gamma}^\rho(x)\, \widetilde{\gamma}^\delta(x)\,, \tag{C16}$$

where ϵ is the fully antisymmetric Levi-Civita tensor, with $\epsilon_{0123} = 1$. We recall that the flat-space $\widetilde{\gamma}^5$ is

$$\gamma^5 = \gamma_5 = \frac{\mathrm{i}}{4!}\, \epsilon_{ABCD} \widetilde{\gamma}^A\, \widetilde{\gamma}^B\, \widetilde{\gamma}^C\, \widetilde{\gamma}^D\,. \tag{C17}$$

For the Schwarzschild geometry, one easily finds

$$\det \overline{g}(x) = -\, w^2 v^6\,, \qquad \sqrt{-\det \overline{g}(x)} = w\, v^3\,, \tag{C18}$$

$$\overline{\gamma}^5(x) = \overline{\gamma}_5(x) = \mathrm{i}\widetilde{\gamma}_0 \widetilde{\gamma}_1 \widetilde{\gamma}_2 \widetilde{\gamma}_3 = \widetilde{\gamma}^5 \equiv \gamma^5\,, \tag{C19}$$

i.e., the flat– and curved–space γ^5 matrices are identical. Variation of the action (C11) gives us

$$\left(\mathrm{i}\overline{\gamma}^\mu\, \nabla_\mu - \gamma^5\, m\right) \psi = 0\,, \tag{C20}$$

which can be rewritten as

$$\mathrm{i}(\overline{\gamma}^0)^2\, \partial_0 \psi = \left(\overline{\gamma}^0\, \overline{\gamma}^i\, p^i + \mathrm{i}\overline{\gamma}^0\, \overline{\gamma}^\mu\, \Gamma_\mu + \gamma^5 m\right)\, \psi\,. \tag{C21}$$

An explicit calculation of the Ricci rotation coefficients show that

$$\overline{\gamma}^0\, \overline{\gamma}^\mu\, \Gamma_\mu = -\frac{\vec{\alpha} \cdot \vec{\nabla} w}{2w^2 v} - \frac{\vec{\alpha} \cdot \vec{\nabla} v}{w v^2}\,, \tag{C22}$$

which is in agreement with Ref. [4, 5], where the $\vec{\alpha} = \widetilde{\gamma}^0 \vec{\widetilde{\gamma}}$ matrices are flat–space matrices. With the help of Eq. (C21), one then finds that $\mathrm{i}\partial_t \psi = H\psi$, where

$$H = \frac{w}{v} \vec{\alpha} \cdot \vec{p} + \frac{\vec{\alpha} \cdot [\vec{p}, w]}{2v} + \frac{w\vec{\alpha} \cdot [\vec{p}, v])}{v^2} + \beta\gamma^5 m\, w\,, \tag{C23}$$

and $\beta = \widetilde{\gamma}^0$. We now stretch the spatial coordinates with the help of the operator $v^{3/2}$, in analogy to the tardyonic case [4, 5], and find the γ^5–Hermitian Hamiltonian

$$H' = v^{3/2}\, H\, v^{-3/2} = \frac{1}{2} \left\{ \vec{\alpha} \cdot \vec{p}, \frac{w}{v} \right\} + \beta\gamma^5\, m\, w\,, \tag{C24}$$

with $H' = \gamma^5\, H'^{+}\, \gamma^5$. Approximating w and v, according to Eq. (C8), to the first order in gravity,

$$w \approx 1 - \frac{r_s}{2r}\,, \quad v \approx 1 + \frac{r_s}{2r}\,, \tag{C25}$$

one finds the tachyonic gravitationally (TG) coupled Hamiltonian

$$H_{\rm TG} = \frac{1}{2} \left\{ \vec{\alpha} \cdot \vec{p}, \left(1 - \frac{r_s}{r}\right) \right\} + \beta\, \gamma^5\, m\, \left(1 - \frac{r_s}{2r}\right)\,. \tag{C26}$$

In the main body of the article, standard notation is exclusively used for the *flat*-space Dirac matrices (no overlining and no tildes), i.e., we denote the $\widetilde{\gamma}^\mu$ as γ^μ.

Appendix D: Higher-Order Terms

As discussed in Secs. II and III, the only difference between the effective high–energy Hamiltonians for tardyons [Eq. (41)] and tachyons [Eq. (54)], derived in the main body of this work, is due the different dispersion relations for (free) tardyons and tachyons, while the gravitational interaction terms are identical to first order in r_s and first order in $1/\mathcal{E}$. This observation can be traced to the fact that the terms multiplying the kinetic operator and the mass in Eqs. (22) and (42), namely, $X = 1 - \frac{r_s}{r}$, and $Y = 1 - \frac{r_s}{2r}$ fulfill the relationship $Y^2/X = 1 + \mathcal{O}(r_s^2)$. One then easily reveals the cancellation mechanism for the terms of first order in r_s by treating X and Y in the non-transformed Hamiltonians (22) and (42) as constants. However, this does not imply that gravitational effects are the same for tardyons and tachyons, in higher orders of G (higher orders of r_s).

The ultrarelativistic decoupling transformation, keeping terms second order in r_s, and to the first order in $1/\mathcal{E}$ (see Appendix B), is expected to lead to differences in the gravitational interaction terms. In the calculation, one needs to take into account the fact that in higher orders of the gravitational coupling constant, we cannot use the starting Hamiltonians as defined in Eqs. (22) and (42). Instead, we must must use higher order approximations to the gravitational terms, which are otherwise neglected in Eq. (C25) (see also Refs. [4–6]). These lead to the initial Hamiltonians

$$H_{\rm ds} = \frac{1}{2} \left\{ \vec{\alpha} \cdot \vec{p}, 1 - \frac{r_s}{r} + \frac{9\, r_s^2}{16 r^2} \right\} + \beta\, m \left(1 - \frac{r_s}{2r} + \frac{r_s^2}{8r^2}\right) \tag{D1}$$

for tardyons and

$$H_{\rm td} = \frac{1}{2} \left\{ \vec{\alpha} \cdot \vec{p}, 1 - \frac{r_s}{r} + \frac{9\, r_s^2}{16 r^2} \right\} + \beta\gamma^5\, m \left(1 - \frac{r_s}{2r} + \frac{r_s^2}{8r^2}\right) \tag{D2}$$

for tachyons. We then transform these Hamiltonians into the Weyl basis using the transform U defined in Eq. (4). Calculations become tedious and lengthy. One observation in generalizing the decoupling transformation is that given a function $f = f(\vec{r})$, then to first order of $1/\mathcal{E}$ one finds

$$\frac{1}{\mathcal{E}}\, f\, \mathcal{E} + \mathcal{E}\, f\, \frac{1}{\mathcal{E}} = 2\, f + \frac{1}{\mathcal{E}} \left[\mathcal{E}, [\mathcal{E}, f]\right] \frac{1}{\mathcal{E}} \to 2\, f\,. \tag{D3}$$

As discussed in Sec. III A, this is due to the fact that the two operators $\mathcal{E}$ act only on the function $f(\vec{r})$, and not on the wavefunction, thus they do *not* give “dominating” energy terms. After three iterations of the transform (per Hamiltonian), one finds for tardyons

$$\mathcal{H}_{\rm ds} = \beta \left(\mathcal{E} + \frac{m^2}{2\mathcal{E}} - \frac{1}{2} \left\{ \mathcal{E}, \frac{r_s}{r} \right\} + \frac{9}{32} \left\{ \mathcal{E}, \frac{r_s^2}{r^2} \right\} - \frac{7\, m^2}{64} \left\{ \frac{1}{\mathcal{E}}, \frac{r_s^2}{r^2} \right\} + \frac{3\, m^2}{16} \frac{r_s}{r} \frac{1}{\mathcal{E}} \frac{r_s}{r} \right)\,, \tag{D4}$$

while for tachyons

$$\mathcal{H}_{\rm tg} = \beta \left(\mathcal{E} - \frac{m^2}{2\mathcal{E}} - \frac{1}{2} \left\{ \mathcal{E}, \frac{r_s}{r} \right\} + \frac{9}{32} \left\{ \mathcal{E}, \frac{r_s^2}{r^2} \right\} + \frac{7\, m^2}{64} \left\{ \frac{1}{\mathcal{E}}, \frac{r_s^2}{r^2} \right\} - \frac{3\, m^2}{16} \frac{r_s}{r} \frac{1}{\mathcal{E}} \frac{r_s}{r} \right)\,. \tag{D5}$$

The final two terms in these Hamiltonians have opposite signs, indicating a difference in the gravitational interaction for tachyons and tardyons.

[1] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[2] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[3] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[4] U. D. Jentschura, Phys. Rev. A **87**, 032101 (2013), [Erratum Phys. Rev. A **87**, 069903(E) (2013)].
[5] U. D. Jentschura and J. H. Noble, Phys. Rev. A **88**, 022121 (2013).
[6] U. D. Jentschura and J. H. Noble, J. Phys. A **47**, 045402 (2014).
[7] E. Eriksen and M. Kolsrud, Supplemento al Nuovo Cimento **18**, 1 (1958).
[8] M. V. Gorbatenko and V. P. Neznamov, Ann. Phys. (Berlin) **526**, 195 (2014).
[9] U. D. Jentschura, Ann. Phys. (Berlin) **526**, A47 (2014).
[10] A. J. Silenko, Phys. Rev. A **77**, 012116 (2008).
[11] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics, Volume 4 of the Course on Theoretical Physics*, 2 ed. (Pergamon Press, Oxford, UK, 1982).
[12] A. Chodos, A. I. Hauser, and V. A. Kostelecky, Phys. Lett. B **150**, 431 (1985).
[13] O. M. P. Bilaniuk, V. K. Deshpande, and E. C. G. Sudarshan, Am. J. Phys. **30**, 718 (1962).
[14] G. Feinberg, Phys. Rev. **159**, 1089 (1967).
[15] M. E. Arons and E. C. G. Sudarshan, Phys. Rev. **173**, 1622 (1968).
[16] O.-M. Bilaniuk and E. C. G. Sudarshan, Nature (London) **223**, 386 (1969).
[17] J. Dhar and E. C. G. Sudarshan, Phys. Rev. **174**, 1808 (1968).
[18] M. Srednicki, *Quantum Field Theory* (Cambridge University Press, Cambridge, 2007).
[19] U. D. Jentschura and B. J. Wundt, Eur. Phys. J. C **72**, 1894 (2012).
[20] U. D. Jentschura and B. J. Wundt, J. Phys. A **45**, 444017 (2012).
[21] J. Ciborowski, Acta Phys. Polon. B **29**, 113 (1998).
[22] T. Chang, *A new Dirac-type equation for tachyonic neutrinos*, e-print hep-th/0011087.
[23] N.-P. Chang, *Oscillations of Faster than Light Majorana Neutrinos: A Causal Field Theory*, e-print hep-ph/0105153.
[24] U. D. Jentschura and B. J. Wundt, J. Phys. G **41**, 075201 (2014).
[25] R. Ehrlich, Phys. Lett. B **493**, 229 (2000).
[26] R. Ehrlich, Astropart. Phys. **35**, 625 (2012).
[27] R. Ehrlich, Astropart. Phys. **41**, 1 (2013).
[28] R. Ehrlich, Astropart. Phys. **66**, 11 (2015).
[29] V. A. Kostelecky and M. Mewes, Phys. Rev. D **85**, 096005 (2012).
[30] U. D. Jentschura, D. Horváth, S. Nagy, I. Nándori, Z. Trócsányi, and B. Ujvári, Int. J. Mod. Phys. E **23**, 1450004 (2014).
[31] W. Pauli, Rev. Mod. Phys. **15**, 175 (1943).
[32] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[33] A. Mostafazadeh, J. Math. Phys. **43**, 205 (2002).
[34] A. Mostafazadeh, J. Math. Phys. **43**, 2814 (2002).
[35] A. Mostafazadeh, J. Math. Phys. **43**, 3944 (2002).
[36] U. D. Jentschura and B. J. Wundt, ISRN High Energy Physics **2013**, 374612 (2013).
[37] D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957).
[38] V. M. Pereira, J. Nilsson, and A. H. Castro Neto, Phys. Rev. Lett. **99**, 166802 (2007).
[39] M. Bojowald, *Canonical Gravity and Applications* (Cambridge University Press, Cambridge, 2011).
[40] O. S. Ivanitskaya, *Extended Lorentz transformations and their applications (in Russian)* (Nauka i Technika, Minsk, USSR, 1969).
[41] O. S. Ivanitskaya, *Lorentzian basis and gravitational effects in Einsteins theory of gravity (in Russian)* (Nauka i Technika, Minsk, USSR, 1969).
[42] D. G. Boulware, Phys. Rev. D **12**, 350 (1975).
[43] M. Soffel, B. Müller, and W. Greiner, J. Phys. A **10**, 551 (1977).
[44] A. K. Gorbatsievich, *Quantum mechanics in general relativity. Basic principles and elementary applications* (Nauka i Technika, Minsk, 1985).
[45] J. Yepez, *Einstein's vierbein field theory of curved space*, e-print arXiv:1106.2037 [gr-qc].
[46] A. Zaloznik and N. S. Mankoc Borstnik, *Kaluza–Klein theory*, advanced seminar 4 at the University of Ljubljana, in the physics department. Available from the URL http://mafija.fmf.uni-lj.si/seminar/files/2011_2012/KaluzaKlein_theo
[47] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, England, 1924).